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FURTHER STUDY ON GEO ENERGY OF GRAPHS

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Abstract: Let $G = (V, E)$ be a (p, q) simple connected graph. K. Palani[7] introduced the concept of geo energy of graph as the sum of absolute values of the spectrum of its Geo matrix. The scope of this study is to inspect numerous bounds on geo energy through graph theory and utilization.

Keywords and Phrases: Geo matrix, Geo spectrum, Geo energy.

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1. Introduction and Preliminaries

The idea of graph energy was debuted by Gutman [3] in 1978 as the sum of absolute values of spectrum of its adjacency matrix. The concept of geodetic was proposed by Harary F [5] in 1993. Let $u, v \in V(G)$. Then $d(u, v)$ is the shortest path of $u - v$ and is known as geodesic path. A set $S \subseteq V(G)$ is a geodetic set if $I[S] = V(G)$, where $I[S]$ is the union of closed intervals $I[u, v]$ consisting of all vertices lying in a $u - v$ geodesic of G, i.e, $I[S] = \bigcup_{u,v \in S} I[u, v]$. The geodetic number of G is the minimum cardinality among all the geo sets and is denoted by $'g'$. K.Palani et. al., [7] introduced the concept of geo energy of graphs and is defines as the sum of absolute values of the spectrum of its Geo matrix. This paper investigate the characteristic of Geo spectrum and Geo energy and derive more bounds on geo energy.

Definition 1.1. *Let $G = (V, E)$ be a (p, q) simple graph. Let $S \subseteq V(G)$ be the minimum geodetic set of G and $|S| = g$. Then the geo matrix of G corresponding to S is a square matrix G_S of order p and is defined as,*

$$G_S = \begin{cases} 1 & for \quad v_i \sim v_j \quad i \neq j \\ 1 & for \quad i = j \quad and \quad v_i \in S \\ 0 & otherwise \end{cases}$$

where the symbol $' \sim '$ denotes the adjacency of the vertices of G. The geo spectrum of the graph G is the eigenvalues of the geo matrix G_S. Let $\lambda_1, \lambda_2, \ldots \lambda_p$ be the spectrum of G_S. Then the geo energy GE of G corresponding to S is defined as, $GE_S(G) = \sum_{i=1}^{p} |\lambda_i|$

Theorem 1.2. [6] *Let a_i and b_i, $1 \leq i \leq p$ are non-negative real numbers, then*

$$\sum_{i=1}^{p}(a_i)^2 \sum_{i=1}^{p}(b_i)^2 - (\sum_{i=1}^{p} a_i b_i)^2 \leq \frac{p^2}{4}(M_1 M_2 - m_1 m_2)^2$$

where $M_1 = max_{1\leq i\leq p}(a_i), M_2 = max_{1\leq i\leq p}(b_i); m_1 = min_{1\leq i\leq p}(a_i)$ and $m_2 = min_{1\leq i\leq p}(b_i)$.

Theorem 1.3. [8] *Suppose a_i and b_i, $1 \leq i \leq p$ are positive real values, then*

$$\sum_{i=1}^{p}(a_i)^2 \sum_{i=1}^{p}(b_i)^2 \leq \frac{1}{4}(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}})^2(\sum_{i=1}^{p} a_i b_i)^2$$

where M_i and m_i are similar to previous theorem.

Lemma 1.4. [9] *Let $a_1, a_2, ..., a_p$ be non negative numbers. Then*

$$p[\frac{1}{p}\sum_{i=1}^{p} a_i - (\prod_{i=1}^{p} a_i)^{\frac{1}{p}}] \leq p\sum_{i=1}^{p} a_i - (\sum_{i=1}^{p}\sqrt{a_i})^2$$
$$\leq p(p-1)[\frac{1}{p}\sum_{i=1}^{p} a_i - ((\prod_{i=1}^{p} a_i)^{\frac{1}{p}}]$$

Theorem 1.5. [2] *Let a_i and b_i, $1 \leq i \leq p$ are positive real numbers, then*

$$\sum_{i=1}^{p}(b_i)^2 + rR\sum_{i=1}^{p}(a_i)^2 \leq (r + R)(\sum_{i=1}^{p} a_i b_i)$$

Where r and R are real constants, $\forall i, 1 \leq i \leq p$ holds $ra_i \leq b_i \leq Ra_i$.

2. Bounds on Geo Energy

Theorem 2.1. *Let $G = (V, E)$ be any simple (p, q) graph. If $\lambda_1, \lambda_2, \ldots \lambda_p$ are the geo spectrum of the matrix $G_S(G)$, then the following condition holds:*
(a) $\sum_{i=1}^{p} \lambda_i = g$.
(b) $\sum_{i=1}^{p} (\lambda_i)^2 = 2q + g$.
Proof. (a) We know that, sum of eigenvalues of $G_S(G)$ is same as the trace of $G_S(G)$. ie, $\sum_{i=1}^{p} \lambda_i = \sum_{i=1}^{p} g_{ii} = |S| = g$.Therefore, $\sum_{i=1}^{p} \lambda_i = g$
(b) Since the sum of squares of the eigenvalues of $G_S(G)$ is the trace of $(G_S(G))^2$,

$$\begin{aligned}\sum_{i=1}^{p} (\lambda_i)^2 &= \sum_{j=1}^{p} \sum_{i=1}^{p} (g_{ij} g_{ji}) \\ &= \sum_{i=1}^{p} (g_{ii})^2 + \sum_{i \neq j} (g_{ij} g_{ji}) \\ &= \sum_{i=1}^{p} (g_{ii})^2 + 2 \sum_{i<j} (g_{ij})^2 = 2q + g\end{aligned}$$

Theorem 2.2. *Let $G = (V, E)$ be any simple (p, q) graph. If $GE(G)$ is a rational number then $GE(G) = \begin{cases} 0 \pmod 2 & if \quad g \quad is\ even \\ 1 \pmod 2 & if \quad g \quad is\ odd \end{cases}$.*
Proof. Let $\lambda_1, \lambda_2, \ldots \lambda_p$ be the geo spectrum of the geo matrix $G_S(G)$. Let $\lambda_1, \lambda_2, ...\lambda_t$ are positive and the rest are of negative sign, then

$$\sum_{i=1}^{p} |\lambda_i| = (\lambda_1 + \lambda_2 + ... + \lambda_t) - (\lambda_{t+1} + \lambda_{t+2} ... + \lambda_p)$$

$$\sum_{i=1}^{p} |\lambda_i| = 2(\lambda_1 + \lambda_2 + ... + \lambda_t) - (\lambda_1 + \lambda_2 + ... + \lambda_p)$$

$$\Longrightarrow GE(G) = 2(\lambda_1 + \lambda_2 + ... + \lambda_t) - \sum_{i=1}^{p} \lambda_i$$

$$\Longrightarrow GE(G) = 2(\lambda_1 + \lambda_2 + ... + \lambda_t) - g, \quad since \quad \sum_{i=1}^{p} \lambda_i = g$$

Hence, $GE(G) = \begin{cases} 0 \pmod 2 & if \quad g \quad is\ even \\ 1 \pmod 2 & if \quad g \quad is\ odd \end{cases}$.

Theorem 2.3. *Let G be a (p,q) graph. Then*

$$GE(G) \geq \sqrt{p(2q+g) - \frac{p^2}{4}(\lambda_1 - \lambda_p)^2}$$

where λ_1 and λ_p are the maximum and minimum absolute values of geo spectrum of G.

Proof. Let $\lambda_1, \lambda_2, ..., \lambda_p$ are the geo spectrum of G. Let $a_i = 1$ and $b_i = |\lambda_i|$ in theorem 1.2, we get,

$$\sum_{i=1}^{p} 1^2 \sum_{i=1}^{p} |\lambda_i|^2 - (\sum_{i=1}^{p} |\lambda_i|)^2 \leq \frac{p^2}{4}(\lambda_1 - \lambda_p)^2$$

By theorem 2.1,

$$p(2q+g) - (GE(G))^2 \leq \frac{p^2}{4}(\lambda_1 - \lambda_p)^2$$

$$GE(G) \geq \sqrt{p(2q+g) - \frac{p^2}{4}(\lambda_1 - \lambda_p)^2}$$

Theorem 2.4. *For any (p,q) connected graph G,*

$$\sqrt{2q+g} \leq GE(G) \leq \sqrt{p(2q+g)}$$

Proof. Consider Cauchy Schwarz inequality,

$$(\sum_{i=1}^{p}(a_i b_i))^2 \leq (\sum_{i=1}^{p}(a_i)^2)(\sum_{i=1}^{p}(b_i)^2)$$

Put $a_i = 1$ and $b_i = |\lambda_i|$ then,

$$(\sum_{i=1}^{p}(|\lambda_i|))^2 \leq (\sum_{i=1}^{p} 1)(\sum_{i=1}^{p}(\lambda_i)^2)$$

$$\Longrightarrow [GE(G)]^2 \leq p\sum_{i=1}^{p}(\lambda_i)^2 = p(2q+g)$$

$$\Longrightarrow GE(G) \leq \sqrt{p(2q+g)} \tag{1}$$

Consider $(GE(G))^2 = (\sum_{i=1}^{p} |\lambda_i|)^2$

$$\geq \sum_{i=1}^{p} |\lambda_i|^2 = 2q+g$$

$$\Longrightarrow GE(G) \geq \sqrt{2q+g} \tag{2}$$

From (1) and (2),

$$\sqrt{2q+g} \leq GE(G) \leq \sqrt{p(2q+g)}.$$

Theorem 2.5. *Let G be a (p,q) simple graph. Let $'\Delta'$ be the absolute value of determinant of the geo matrix G_S of G. Then,*

$$\sqrt{(2q+g)+p(p-1)\Delta^{\frac{2}{p}}} \leq GE(G) \leq \sqrt{(2q+g)(p-1)+\Delta^{\frac{2}{p}}}$$

Proof. Let $X = p[\frac{1}{p}\sum_{i=1}^{p}\lambda_i^2 - (\prod_{i=1}^{p}\lambda_i^2)^{\frac{1}{p}}]$

$$= p[\frac{1}{p}(2q+g) - (\prod_{i=1}^{p}|\lambda_i|)^{\frac{2}{p}}] = (2q+g) - p\Delta^{\frac{2}{p}}$$

By lemma 1.4, if $a_i = \lambda_i^2, i = 1, 2, \ldots, p$; then,

$$X \leq p\sum_{i=1}^{p}\lambda_i^2 - (\sum_{i=1}^{p}|\lambda_i|)^2 \leq (p-1)X$$
$$\Longrightarrow X \leq p(2q+g) - (GE(G))^2 \leq (p-1)X$$

Simplifying this we get,

$$\sqrt{(2q+g)+p(p-1)\Delta^{\frac{2}{p}}} \leq GE(G) \leq \sqrt{(2q+g)(p-1)+\Delta^{\frac{2}{p}}}.$$

Theorem 2.6. *Suppose the geo spectrum of $G = (p,q)$ consists of non-zero elements, then $GE(G) \geq \frac{2\sqrt{\lambda_1\lambda_p}\sqrt{p(2q+g)}}{\lambda_1+\lambda_p}$, where λ_1 and λ_p are the minimum and maximum values of $|\lambda_i|'s$.*

Proof. Let $\lambda_1, \lambda_2, \ldots \lambda_p$ are the geo spectrum of G. Put $a_i = |\lambda_i|, b_i = 1$ in theorem 1.3, we get

$$\sum_{i=1}^{p}|\lambda_i|^2\sum_{i=1}^{p}1^2 \leq \frac{1}{4}(\sqrt{\frac{\lambda_p}{\lambda_1}}+\sqrt{\frac{\lambda_1}{\lambda_p}})^2(\sum_{i=1}^{p}|\lambda_i|)^2$$
$$p(2q+g) \leq \frac{1}{4}(\frac{(\lambda_1+\lambda_p)^2}{\lambda_1\lambda_p})(GE(G))^2$$
$$GE(G) \geq \frac{2\sqrt{\lambda_1\lambda_p}\sqrt{p(2q+g)}}{\lambda_1+\lambda_p}$$

Theorem 2.7. *Let $\lambda_1, \geq \lambda_2 \geq \ldots \geq \lambda_p$ be the non-increasing sequence of geo spectrum of the graph $G = (p,q)$. Then $GE(G) \geq \frac{p|\lambda_1||\lambda_p|+(2q+g)}{|\lambda_1|+|\lambda_p|}$, where λ_1 and λ_p*

are the minimum and maximum values of $|\lambda_i|'s$.

Proof. Let $\lambda_1, \lambda_2, ...\lambda_p$ are the geo spectrum of G. Put $a_i = 1, b_i = |\lambda_i|, r = |\lambda_p|$ and $R = |\lambda_1|$ in theorem 1.5, we get

$$\sum_{i=1}^{p} |\lambda_i|^2 + |\lambda_1||\lambda_p| \sum_{i=1}^{p} 1 \leq (|\lambda_1| + |\lambda_p|) \sum_{i=1}^{p} |\lambda_i|$$

Now substitute the values of $\sum_{i=1}^{p} |\lambda_i|^2$ and $\sum_{i=1}^{p} |\lambda_i|$ from theorem 2.1, we get

$$GE(G) \geq \frac{p|\lambda_1||\lambda_p| + (2q + g)}{|\lambda_1| + |\lambda_p|}$$

3. Conclusion

This paper investigates the bounds on geo energy of any simple connected graph. For such graphs all the above inequalities holds.

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References

[1] Buckley F., Harary F., Distance in Graphs, Addison - Wesly Publishing Company, Redwood City, CA, 1990.

[2] Diaz J. B. and Metcalf F. T., Stronger forms of a class of inequalities of G. Pólya-G.Szeg″o and L. V. Kantorovich, Bulletin of the AMS - American Mathematical Society, 69 (1963), 415-418.

[3] Gutman I., The energy of a graph, Ber. Math. Stat. Sekt. For schungszent, Graz, 103 (1978), 1–22.

[4] Gutman I., The energy of a graph: old and new results, Algebraic Combinatorics and Applications (Gößweinstein, 1999), Springer, Berlin, (2001), 196–211.

[5] Harary F., Loukakis E., Tsouros C., The Geodetic Number of a Graph, Mathl. Comput. Modelling, Vol. 17, No. 11 (1993), 89-95.

[6] Ozeki N., On the estimation of inequalities by maximum and minimum values, Journal of College Arts and Science, Chiba University, 5 (1968), 199-203. (in Japanese)

[7] Palani K., Lalitha Kumari M., Locating geo spectrum snd geo energy of grsphs, Adv Appl Math Sci, 21 (4) (2022), 2157-2167.

[8] Pólya G. and Szeg″o G., Problems and Theorems in analysis. Series, Integral Calculus, Theory of Functions, Springer, Berlin, 1972.

[9] Zhou B., Gutman I., Aleksic T., A note on the Laplacian energy of graphs, MATCH Commun. Math. Chem, 60 (2008), 441-446.